1.0

1.1

1.25 1.4 1.6

2.8 2.5
3.2 2.2
36
40 2.0

1.8

MICROCOPY RESOLUTION TEST CHART
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MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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EQUATORIAL MOTION ABOUT AN OBLATE PRIMARY

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ABSTRACT

This Report discusses motion in the equatorial plane of an oblate primary. We include terms due to non-sphericity through J_2 in the potential. This problem is exactly soluble in the same sense that the ordinary two-body problem is exactly soluble. The full solution is given, the connection with perturbation theory (as $J_2 \neq 0$) is demonstrated, and the stability of circular orbits is discussed. The analytical solution involves all three incomplete elliptic integrals which limits its transparency.

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I. INTRODUCTION

We investigate herein some aspects of the motion of a satellite about an oblate primary. We restrict the problem in two ways. First we only consider motion in the equatorial plane. Second we allow departures from sphericity to be manifest only in a J_2 term [see Eq. (1) below]. These two restrictions yield an analytically tractable problem in that the orbit and the time dependence of the location may be explicitly found. Even with these two restrictions the problem is still important within the solar system - most natural satellites, be they moons or rings, lie in the equatorial plane of their primaries and many artificial satellites of the Earth lie in the plane of the celestial equator.

The principal advantage of being able to solve a problem analytically is that one then may approximate the solution from a position of strength. For example expanding sinx in a power series in x can be rigorously justified, a radius of convergence computed, and with luck or skill, a truncation error established. Here one's mathematical underpinnings are secure. In contrast solving for $y(x)$ in

$$\frac{d^2y(x)}{dx^2} = -y(x) \quad ; \quad y(0) = 0 \ , \ y'(0) = 1$$

by an approximation technique rarely offers such a complete set of luxuries. That the solution of this differential equation is $y(x) = $ sinx is of no help unless one knows that - precisely when approximation is superfluous.

We already know how to approximately solve the general oblate primary problem via Lagrange's planetary equations. Here we shall derive some of these results from the exact solution. Moreover, we shall exploit other

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approximation techniques to investigate the orbit and the time dependence.

It turns out that a third order polynomial in the distance is critical to determining the nature of the solution. When $J_2 = 0$ the cubic has zero for one of its roots and the solution for the motion is accomplished in terms of trigonometric functions. For $J_2 \neq 0$ the full cubic must be considered and the solution involves elliptic functions. Below we formulate the problem, develop the central role of this cubic, and analyze its properties. The following sections discuss the orbit, the time dependence of the longitude, and the time dependence of the distance. As appropriate approximations are used to rederive the classical results referred to above. Finally, because J_2 is small in all practical applications, the notation of the Keplerian two-body problem has been maintained. Of course for $J_2 \neq 0$ a is not the semi-major axis nor is e the eccentricity. However, as $J_2 \rightarrow 0$ the quantities denoted here by a, e, etc. do approach their Keplerian values.

II. FORMULATION

A. Physics and Geometry

We choose as the origin the center of mass of the oblate primary. The equatorial plane of the primary coincides with the plane z = 0. Since the primary is a spheroid there is no preferred origin of longitudes. Cartesian coordinates x, y, and z are related to spherical coordinates r, θ, φ (distance, latitude, and longitude) by

$$x = r \cos\theta \cos\phi$$

$$y = r \cos\theta \sin\phi$$

$$z = r \sin\theta$$

We take this coordinate system to be an inertial one. The gravitational potential of the oblate primary, whose total mass is M, is assumed to be

$$U(\underline{r}) = - \frac{GM}{r} [1 + (J_2/2r^2)(1 - 3 \sin^2\theta)] \tag{1}$$

where G is the universal constant of gravitation and $r = |\underline{r}|$. The physical interpretation of J_2 is that it is the difference between the polar and the equatorial moments of inertia per unit mass. $J_2 \geq 0$ herein (if J_2 were negative then we would be describing prolate spheroids). Note too that the dimensions of J_2 are those of a length squared.

The equations of motion of a particle are

$$\ddot{\underline{r}} = - \nabla U(\underline{r}) \tag{2}$$

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where

$$\underline{\ddot{r}} = (\ddot{r}-r\dot{\phi}^2\cos^2\theta-r\dot{\theta}^2, \ -r\ddot{\theta}-2\dot{r}\dot{\theta}-r\dot{\phi}^2\sin\theta\cos\theta, \ (r\ddot{\phi}+2\dot{r}\dot{\phi})\cos\theta-2r\dot{\theta}\dot{\phi}\sin\theta)$$

and

$$\nabla U(\underline{r}) = \left(\frac{\partial U}{\partial r}, \ \frac{-1}{r}\ \frac{\partial U}{\partial\theta}, \ \frac{\sec\theta}{r}\ \frac{\partial U}{\partial\phi}\right)$$

Now $\partial U/\partial\theta = 0$ at $\theta = 0$ so if at some arbitrary instant $t = T$ θ and $\dot\theta$ vanish, then so does $\ddot\theta$ as well as all higher time derivatives of θ. There-fore, with the initial conditions $\theta(T) = \dot\theta(T) = 0$, $\theta(t) = 0$ $\forall t$. With this simplification the equations of motion (2) reduce to (since $\partial U/\partial\phi = 0$)

$$\ddot{r} - \dot{r}\phi^2 = -\frac{\partial U}{\partial r}\bigg|_{\theta=0}$$

$$r\ddot\phi + 2\dot r\dot\phi = 0 \qquad \text{or} \qquad \frac{d}{dt}\ (r^2\dot\phi) = 0 \tag{3}$$

As in the Keplerian two-body problem define the constant L by

$$L = r^2\dot\phi \tag{4a}$$

and rewrite the radial equation of motion as

$$\ddot{r} - L^2/r^3 = -\ \frac{\partial U}{\partial r}\bigg|_{\theta=0}$$

This is directly integrable to

$$\frac{\dot{r}^2}{2} = E - U - L^2/(2r^2) \tag{4b}$$

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where E is a constant of integration easily identified with the total energy per unit mass.

Instead of dealing with E and L we choose a different parametrization. Define (μ = GM) a and e via

$$E = {}^{-\mu}/2a \quad , \quad L^2 = \mu a(1-e^2) \tag{5}$$

Then since $U|_{\theta=0} = -(\mu/r)[1 + J_2/(2r^2)]$ the conservation of energy equation may be written as

$$\dot{r}^2 = \frac{-\mu F(r)}{ar^3} \tag{6}$$

where

$$F(r) \equiv r^3 - 2ar^2 + a^2(1-e^2)r - aJ_2$$

Note that for bound motion E<0 so a>0. We also can adjust the handedness of our coordinate system such that $L \geq 0$, therefore $1 \geq e^2$. This is all we can deduce, in general, about the ranges of a and e. However, since the physically meaningful constants of integration are E and L, our parametrization in terms of a, e is valid only if the Jacobian $\partial(E,L)/\partial(a,e) \neq 0$. This implies $a < \infty$ and $e \neq 0$.

B. The Cubic F(r)

We can write F(r) in a variety of forms,

$$F(r) = r^3 - 2ar^2 + a^2(1-e^2)r - aJ_2$$

$$= r[r^2 - 2ar + a^2(1-e^2)] - aJ_2$$

$$= r[r - a(1+e)][r-a(1-e)] - aJ_2$$

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$$= r(r - r_+)(r - r_-) - aJ_2$$

$$= (r - R_a)(r - R_p)(r - R_0) \tag{7}$$

where $r_\pm = a(1 \pm e)$ and R_a, R_p, R_0 are the three roots of F. Were $J_2 = 0$ then the three roots of F would be r_+, r_-, and 0. Since if J_2 vanishes a and e are the semi-major axis and eccentricity it follows that when $J_2 = 0$ r_+ and r_- are the maximum and minimum distances of the particle from the primary. The labeling of the three roots of F when $J_2 \neq 0$ is suggestive of this and we assume $R_a \geq R_p \geq R_0$. We first show that F has three real roots for small J_2.

The discriminant of the cubic $z^3 + a_1 z^2 + a_2 z + a_0$ is given by $Q^3 + R^2$ where

$$Q = a_{1/3} - (a_{2/3})^2 \qquad , \qquad R = (a_1 a_2 - 3a_0)/6 - (a_{2/3})^3$$

Here $a_2 = -2a$, $a_1 = a^2(1-e^2)$, and $a_0 = -aJ_2$ so

$$Q^3 + R^2 = \frac{-a^6 e^2(1-e^2)^2}{27} + \frac{a^4 J_2(-1+9e^2)}{27} + \frac{a^2 J_2^2}{4} \tag{8}$$

For small enough $J_2(>0)$ this is negative. When the discriminant of a cubic is negative then the cubic has three real, unequal roots. If J_2 increases from 0 until $Q^3 + R^2 = 0$ then the cubic has three real roots and at least two of them are equal. This corresponds to the case of a circular orbit and will be discussed below.

The special case of $e=0$, $J_2 \neq 0$ implies that F has the form $F(r) = r(r-a)^2 - aJ_2$. In this case we would expect F to have three real roots near a, a, and 0. Directly from the analytical solution of a cubic

for small J_2, we may show that the roots are approximately equal to $a + \sqrt{J_2}$, $a - \sqrt{J_2}$, and 0 in this case. In the second order of approximation we find (from Newton's method) that the roots are approximately equal to $a \pm \sqrt{J_2}$ - $J_2/(2a)$, J_2/a. The special case of $J_2 = 0$, $e \neq 0$ has for the roots $r_\pm$, 0. Further specializing to $e{=}0$ we find that the roots are a, a, and 0.

Returning to the general problem we assume J_2 is small enough that $Q^3 + R^2 < 0$. While we may obtain the values of R_a, R_p, and R_0 directly from the analytical solution for a cubic this yields nothing transparent. Instead let us compute approximate values for these quantities by exploiting the fact that as $J_2{\to}0$, R_a, R_p, and R_0 must approach r_+, r_-, and 0. Using Newton's method we find that the first order approximations to the roots of F are

$$R_a \simeq r_+ + J_2/(2er_+) > r_+$$

$$R_p \simeq r_- - J_2/(2er_-) < r_-$$

$$R_0 \simeq 0 + J_2/p \qquad > 0 \qquad\qquad (9)$$

where $p = a(1-e^2)$. Continuing the iteration process of Newton's method we find, to second order in J_2 but not to second order in the iteration scheme,

$$R_a \simeq r_+ + J_2/(2er_+) - J_2^2(1+3e)/(2er_+)^3$$

$$R_p \simeq r_- - J_2/(2er_-) + J_2^2(1-3e)/(2er_-)^3$$

$$R_0 \simeq 0 + J_2/p + 2J_2^2/p^3 \qquad\qquad (10)$$

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Note that the $J_2 \neq 0$, $e = 0$ results cannot be obtained from these formulas merely by setting $e = 0$. From Eqs. (9, 10) we conjecture that

$$R_a \geq r_+ \geq r_- \geq R_p \geq R_0 \geq 0$$

To further elucidate the nature of $F(r)$ we compute dF/dr and solve for its roots. The results are that dF/dr vanishes if $r = R_\pm$; $R_\pm = (a/3) [2 \pm (1+3e^2)^{1/2}]$. From the values of d^2F/dr^2 at these two points we deduce that at R_+ F has a local minimum while at R_- F has a local maximum. Since $F(\pm\infty) = \pm \infty$ and $F(0) = -aJ_2 < 0$ we further conjecture that

$$R_a \geq r_+ \geq R_+ \geq r_- \geq R_p \geq R_- \geq R_0 \geq 0$$

Figure 1 contains $F(r)$ for $a = 1$, $e = 1/3$, $J_2 = 1/10$. The numerical values of R_0, R_-, R_p, r_-, R_+, r_+, and R_a are 0.176200992, 0.281766487, 0.398063916, 0.666666667, 1.051566846, 1.333333333, and 1.425735091 in this case. Note that for the extrema of F to be real we must have $e^2 \geq -1/3$.

The results of this analysis, remembering Eq. (6), is that the motion is confined to that range of r such that $F(r) \leq 0$, i.e. $r \epsilon [R_p, R_a]$. When $r = R_p$ or R_a, the radial velocity vanishes and these represent the turning points of the motion. Lastly, from the different forms of F in Eq. (7) we see that

$$R_a + R_p + R_0 = 2a = r_+ + r_-$$

$$R_a R_p + R_a R_0 + R_p R_0 = a^2(1-e^2) = r_+ r_- \tag{11}$$
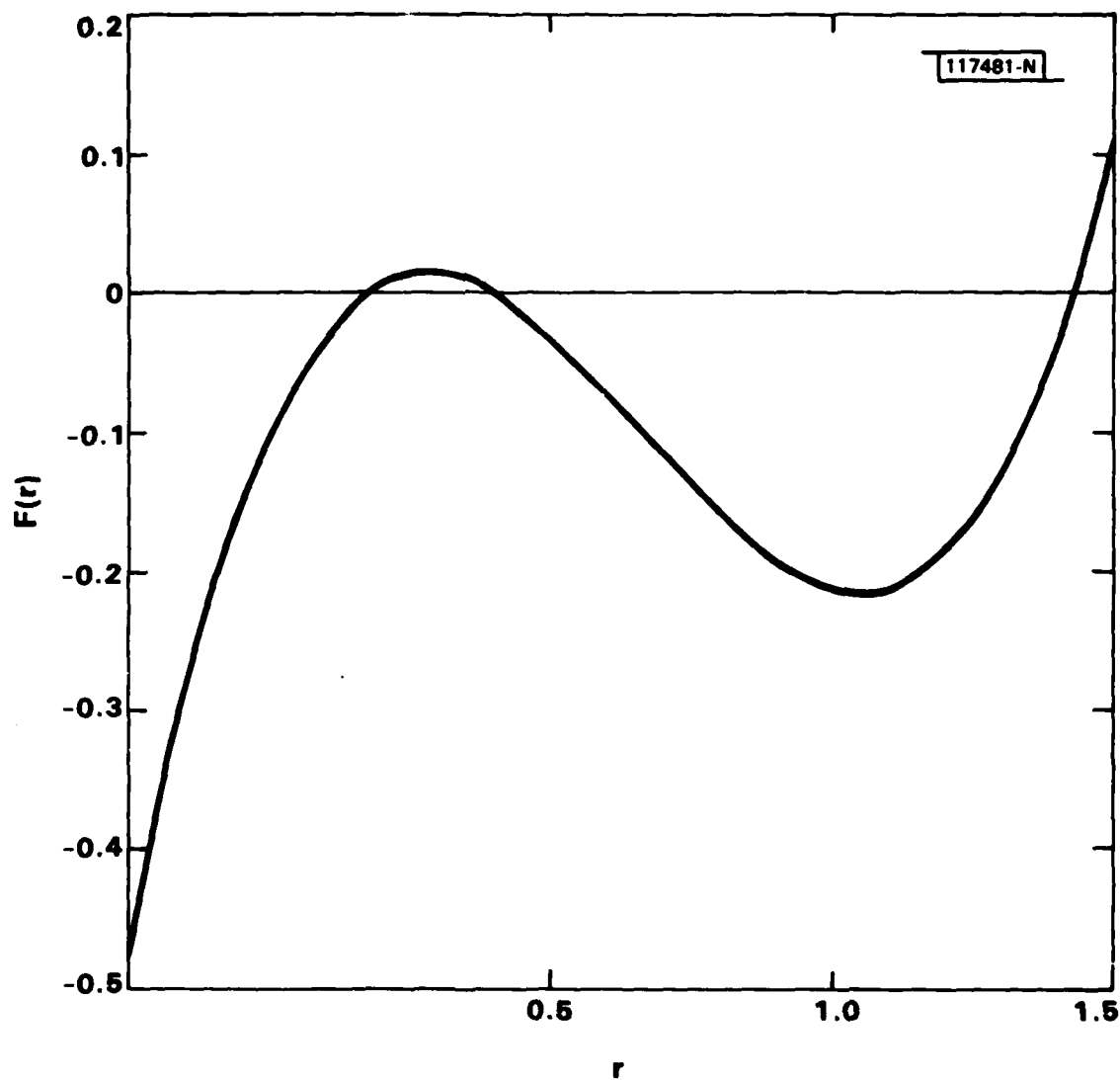
$$R_a R_p R_0 = aJ_2$$

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Fig. 1   F(r) near r = 0 for a = 1, e = 1/3, J_2 = 1/10.

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Having gleaned the maximum amount of information concerning F and dr/dt we now turn to a discussion of the orbit, $r(\phi)$.

III. THE ORBIT

A. The Exact Solution

From the definition of L in Eq. (4a) and the formula for $\dot{r}$ in Eq. (6) we may derive that $r'= dr/d\phi$ is related to $\dot{r} = dr/dt$ via

$$\dot{r} = Lr'/r^2$$

Therefore, the equation of the orbit is $[p = a(1-e^2)]$

$$(r')^2 = \frac{-rF(r)}{ap} \tag{12}$$

Orient the xy axes (arbitrary because the primary is a spheroid) such that at $t=T$ $r=R_p$ and $\phi=0$. Suppose that the earliest time when $r=R_a$ is $t = T + t_a$ where $\phi = \phi_a$. Let the next time when $r = R_p$ be $t = T + t_p$ where $\phi = \phi_p$. Note that since the differential equation of the orbit is even in ϕ we can build up the orbit by reflecting about $\phi = 0$ either one of these segments. Since we know that $R_p \leq r \leq R_a$ and that $\dot{r}^2$ and $(r')^2$ are non-negative it follows that $\dot{r}$ and r' are > 0 for $r\epsilon(R_p,R_a)$, $t - T\epsilon(0,t_a)$ and that $\dot{r}$ and r' are < 0 for $r\epsilon(R_a,R_p)$, $t - T\epsilon(t_a,t_p)$. Both $\dot{r}$ and r' vanish if $r = R_a$ or R_p. Hence, the solution of Eq. (12) is

$$\int_{R_p}^{r} \frac{ds}{[-sF(s)]^{1/2}} = \frac{+1}{(ap)^{1/2}} \int_{0}^{\phi} d\psi \qquad \phi\epsilon[0,\phi_a], \ t - T\epsilon[0,t_a]$$

$$\int_{R_a}^{r} \frac{ds}{[-sF(s)]^{1/2}} = \frac{-1}{(ap)^{1/2}} \int_{\phi_a}^{\phi} d\psi \qquad \phi\epsilon[\phi_a,\phi_p], \ t - T\epsilon[t_a,t_p]$$

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Define the angles λ and Λ by

$$\sin^2\lambda = \frac{R_a - R_0}{R_a - R_p} \cdot \frac{r - R_p}{r - R_0} \quad , \quad \sin^2\Lambda = \frac{R_p}{R_a - R_p} \cdot \frac{R_a - r}{r} \tag{13}$$

and the modulus $k(>0)$ by

$$k^2 = \frac{R_a - R_p}{R_a - R_0} \frac{R_0}{R_p} \quad , \qquad k^2 \epsilon [0,1] \tag{14}$$

Then the results of the above two integrations are

$$\left[\frac{(R_a - R_0)R_p}{ap} \right]^{\frac{1}{2}} \phi = 2F(\lambda,k)$$

$$\left[\frac{(R_a - R_0)R_p}{ap} \right]^{\frac{1}{2}} (\phi - \phi_a) = 2F(\Lambda,k)$$

where F is the incomplete elliptic integral of the first kind. If we introduce $\gamma(>0)$ by (note as $J_2 \rightarrow 0$, $k^2 \rightarrow 0$, $\gamma^2 \rightarrow 1$)

$$\gamma^2 = \frac{(R_a - R_0)R_p}{ap} = \frac{(R_a - R_0)R_p}{(R_a + R_0)R_p + R_a R_0}$$

then these may be more compactly written as

$$\gamma\phi = 2F(\lambda,k) \quad \phi \epsilon [0, \phi_a]$$

$$\gamma(\phi - \phi_a) = 2F(\Lambda,k) \quad \phi \epsilon [\phi_a, \phi_p] \tag{15}$$

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We can now show that $\phi_p = 2\phi_a$. To see this observe that if $r = R_a$ $\lambda = \pi/2$ so $\gamma\phi_a = 2K(k)$ where K is the complete elliptic integral of the first kind. Also we note that if $r = R_p$ $\Lambda = \pi/2$ so $\gamma(\phi_p - \phi_a) = 2K(k)$ too or $\phi_p = 2\phi_a = 4K(k)/\gamma$.

Equation (15) gives us $\phi(r)$ explicitly. From the standard discussions of the Keplerian two-body problem we are used to dealing with the inverse of this, $r(\phi)$. Since
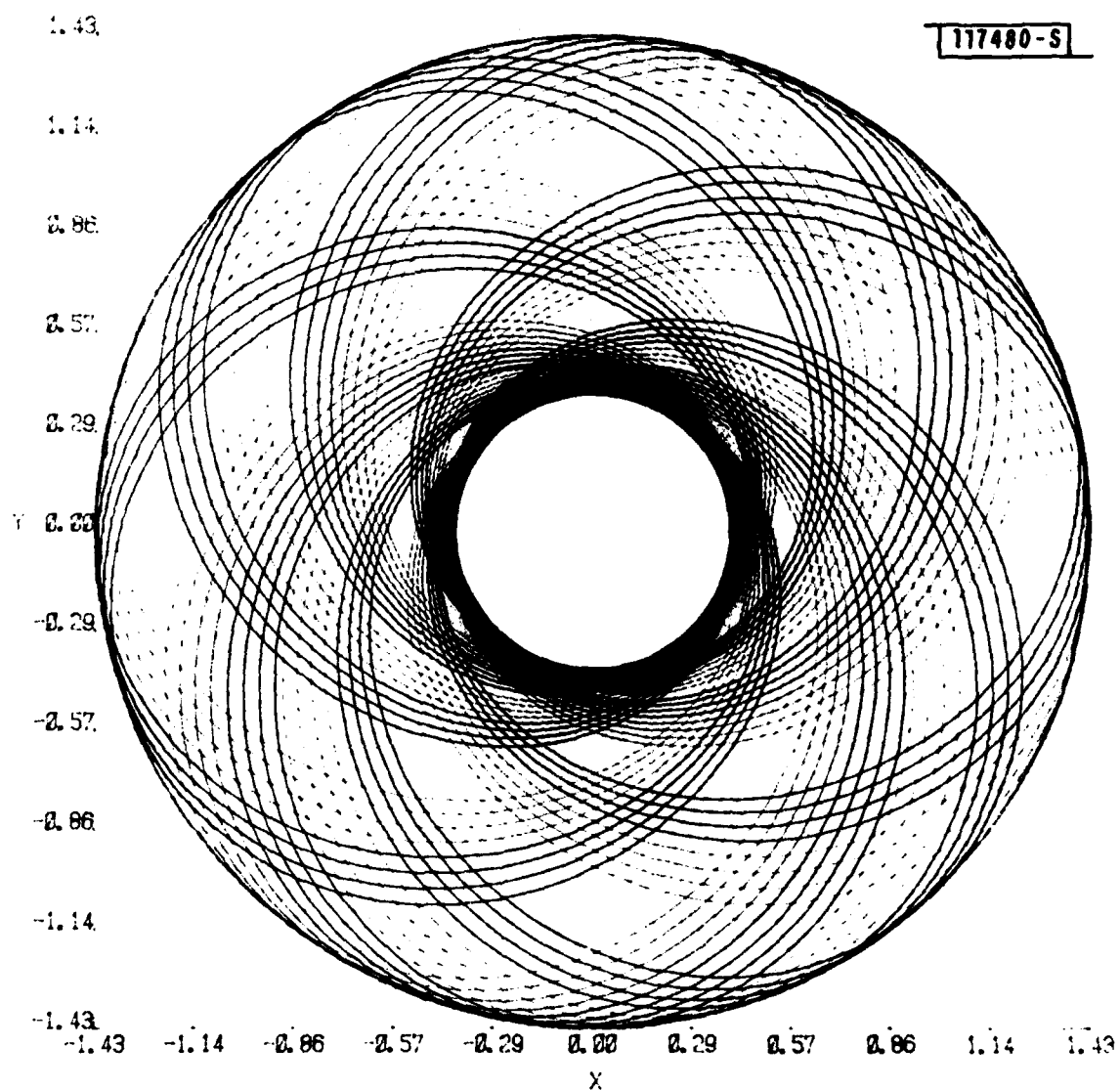
$$sn^{-1}(\sin\psi,\kappa) = F(\psi,\kappa)$$

where $sn(\psi,\kappa)$ is the sine amplitude function of argument ψ, modulus κ, we deduce from Eq. (15), with $u = \gamma\phi/2$, that

$$r(\phi) = \frac{R_p[1-k^2 sn^2(u,k)]}{1-m sn^2(u,k)} = \frac{R_p dn^2(u,k)}{1-m sn^2(u,k)} \tag{16}$$

where $dn(\psi,\kappa)$ is the delta amplitude function of argument ψ, modulus κ and m is an abbreviation for

$$m = \frac{R_a - R_p}{R_a - R_0} = (R_p/R_0)k^2, \quad 1 \geq m \geq k^2 \tag{17}$$

Equation (16) was derived from the upper form in Eq. (15). Had we used the lower form and the addition theorem for $sn(\psi,\kappa)$, the identical result would've been obtained. This demonstrates the analytical equivalence of the two results and that Eq. (16) holds over the entire orbit. It is clear that the orbit is periodic because the elliptic functions are periodic; $sn(\psi,\kappa)$ and $dn(\psi,\kappa)$ share $4K(\kappa)$ as a period. The orbit doesn't repeat however as can be clearly seen in Fig. 2. This is for $a = 1$, $e = 1/3$, $J_2 = 1/10$ and shows $r(\phi)$

ORBIT FOR 64 PERIODS

Fig. 2 64 orbits for a = 1, e = 1/3, J_2 = 1/10. The argument of periapse advances by almost 180° per revolution yielding the double-lobed structure.

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for 64 complete revolutions. The original plot was in three colors, which can be seen in the gradations of grey. The values of k, m, γ and ϕ_a in this case are 0.603365954, 0.822443482, 0.748043275, and 268°59733.

B. The Advance of the Periapse

A well known result from perturbation theory, via Lagrange's equations of motion, is that there is a secular advance of the argument of periapse. From Eq. (16) we see that $r(0) = r(\phi_p) = r(2\phi_a)$ so that this advance is evenly distributed over each half of the orbit. Its amount is $\Delta\omega = \phi_p - 2\pi = 4K(k)/\gamma - 2\pi$. To first order in J_2

$$k^2 = 2eJ_2/p^2 \ , \quad \gamma^2 = 1 - J_2(3-e)/p^2 \tag{18}$$

and as $k^2 \to 0$ $K(k) \to (\pi/2)(1 + k^2/4)$. Therefore we calculate

$$\Delta\omega = 3\pi J_2/p^2$$

confirming the perturbation theory value.

C. An Approximate Orbit

As $k^2 \to 0$

$$sn(\psi,\kappa) \to \sin\psi - (\kappa/2)^2(\psi - \sin\psi\cos\psi)\cos\psi$$

$$dn(\psi,\kappa) \to 1 - 2(\kappa/2)^2\sin^2\psi$$

Utilizing these approximations and those in Eq. (18) in Eq. (16) we are led to, after some algebra,

$$r(\phi) \to \frac{p}{1+e\cos\phi} \left\{ 1 - \frac{J_2[\cos\phi + 3e + 3e^2(\cos\phi + \phi\sin\phi) + e^3(1+\sin^2\phi)]}{2ep^2(1+e\cos\phi)} \right\} \tag{19}$$

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The first term is just the Keplerian orbit. We also note that, to first order in J_2, $r(0) = r(2\pi) = R_p$ and $r(\pi) = R_a$. We can recover the value of $\Delta\omega$ by remembering that at an apse $r' = 0$. Computing this from Eq. (19) and assuming $\phi_a = \pi + \Delta\omega/2$ or $\phi_p = 2\pi + \Delta\omega$, where $\Delta\omega$ is of order J_2, we immediately recover the above value for $\Delta\omega$.

D. An Alternative Derivation

The approximate form of $r(\phi)$ in Eq. (19) can be derived directly from the differential equation (12). Set

$$r(\phi) = r_0(\phi) + r_1(\phi)$$

where $r_0(\phi)$ is the Keplerian solution $p/(1 + e\cos\phi)$ and r_1 is of order J_2. Using this in Eq. (12) and linearizing in r_1 yields

$$r_1' + fr_1 = J_2 g/a^2$$

where

$$f(\phi) = r_0 \, [2r_0^2 - 3ar_0 + a^2(1-e^2)]/(apr_0')$$

$$g(\phi) = a^2 r_0/(2pr_0')$$

The general solution for $r_1(\phi)$ is

$$r_1(\phi) = C \exp[-\int^\phi f(\phi')d\phi'] + (J_2/a^2)\exp[-\int^\phi f(\phi')d\phi']\int^\phi g(\phi') \cdot$$

$$\exp[\int^{\phi'} f(\phi'')d\phi'']d\phi'$$

where C is the arbitrary constant of integration. After performing the integrals we find that

$$r_1(\phi) = \frac{C\sin\phi}{(1+e\cos\phi)^2} - \frac{J_2[\cos\phi + 3e + 3e^2(\cos\phi + \phi\sin\phi) + e^3(1+\sin^2\phi)]}{2ep(1+e\cos\phi)^2}$$

We set $C = 0$ because we don't need it to reproduce $r(0) = r(2\pi) = R_p$ or $r(\pi) = R_a$ to first order in J_2. The result is identical to that obtained above from the expansion of the exact solution, Eq. (19).

E. A Precessing Ellipse

Define a pseudo semi-major axis and eccentricity α and ε, via

$$2\alpha = R_a + R_p$$
$$2\alpha\varepsilon = R_a - R_p \qquad (20)$$

To first order in J_2 they are given by

$$\alpha = a - J_2/(2p)$$

$$\varepsilon = e + J_2(1+e^2)/(2aep)$$

The function

$$r(\phi) = \frac{\alpha(1-\varepsilon^2)}{1+\varepsilon\cos\gamma\phi}$$

represents a precessing ellipse of semi-major axis α and eccentricity ε. The argument of periapse advances by $2\pi/\gamma - 2\pi$ which falls short of the actual value by $\pi k^2/2\gamma$ as $J_2 \to 0$. Hence, although this form is intuitively attractive and correctly indicates the limits of the motion, it can't

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adequately replace Eq. (19) as an approximation to the orbit-even in lowest order. In particular it reproduces all of Eq. (19) except for the e^3 term which is now $e^3(1-\phi\sin\phi)$ instead of $e^3(1-\sin^2\phi)$.

F. Circular Orbits

If the orbit is a circle then r is a constant. From Eq. (16) this implies that $dn^2(u,k)$ is proportional to $1-msn^2(u,k) \forall u$. As $dn^2(\psi,\kappa) = 1-\kappa^2 sn^2(\psi,\kappa)$ this is possible if and only if the proportionality constant is unity and $m=k^2$. For m to be equal to k^2 then (i) $k^2=0$, $m=0$ and $R_a = R_p$ or (ii) $k^2=1$, $m=1$ and $R_p = R_0$. In either case F(r) has a double root. It is a simple matter to show from Eq. (6) that if $F(r) = (r-R_1)^2(r-R_2)$ and one applies a small perturbation to the orbit $r=R_1$, then for this orbit to be stable it must be that $R_1 > R_2$. As unstable orbits are of no interest in this discussion it follows that the interesting case is (i) with $r = R_a = R_p > 0$.

From the analytical solution of a cubic equation we know that a cubic has a double root when its discriminant vanishes. From Eq. (8) we see that forcing $Q^3 + R^2$ to vanish implies a relationship between J_2, a, and e. Since J_2 is a given of the problem and circularity is a statement concerning angular momentum rather than energy, it's really a relationship for $e(J_2,a)$. As $Q^3 + R^2 = 0$ is a cubic in e^2 but a quadratic in J_2, we'll deal with $J_2(a,e)$ here. One finds that a vanishing discriminant implies

$$27J_2/(2a^2) = 1-9e^2 \pm (1+3e^2)^{3/2}$$

Since J_2 is real and non-negative we must have $e^2 \geq -1/3$ (cf above). Since L is real and positive we must have $1 \geq e^2$. If we take the upper sign then J_2 is always ≥ 0 for $e^2 \epsilon[-1/3, 1]$ while if we take the lower sign then J_2 is ≥ 0 only for $e^2 \epsilon[-1/3,0]$.

If we actually solve for the roots of $F(r) = 0$ under the constraint $Q^3 + R^2 = 0$ we find that they are $(a/3)[2 \pm 2(1 + 3e^2)^{1/2}]$, $(a/3)[2 \mp (1 + 3e^2)^{1/2}]$ (twice) with the signs the same order as above. Our stability analysis demonstrated that the physically interesting solution demanded that the double root be the larger. This occurs for the lower sign and

$$R_0 = (2a/3)[1 - (1+3e^2)^{1/2}]$$

$$R_a = R_p = (a/3)[2 + (1+3e^2)^{1/2}] \quad \text{with } e^2 \epsilon [-1/3, 0]$$

Note that if $e^2 = -1/3$ then F has a triple root at $2a/3$ and $J_2 = 8a^2/27$. Figure 3 shows F for $e^2 = -1/9$, $a = 1$, and $J_2 = 0.107827329$. Note also that if $e^2 = 0$, $J_2 = 0$ then we return to our familiar meaning of a circular orbit as one with zero eccentricity. Remember then if $J_2 = 0$ then a circular orbit is the one with maximum angular momentum. Here, with $J_2 \neq 0$, again the angular momentum is larger for a circular orbit (because $e^2 \leq 0$) than it is for a non-circular one ($e^2 \geq 0$). The triple root represents an unstable orbit but has the maximum angular momentum.

A last note: Once J_2 is large enough, $Q^3 + R^2 > 0 \quad \forall e^2 \epsilon [-1/3, 1]$ and F has a single real root and a pair of complex conjugate roots. The single real root represents a circular orbit. This happens for $\frac{a^2}{10} < J_2 < \frac{a^2}{2}$. As $J_2 \to \infty$ the radius of this unstable orbit $\to (aJ_2)^{1/3} + 2a/3$.
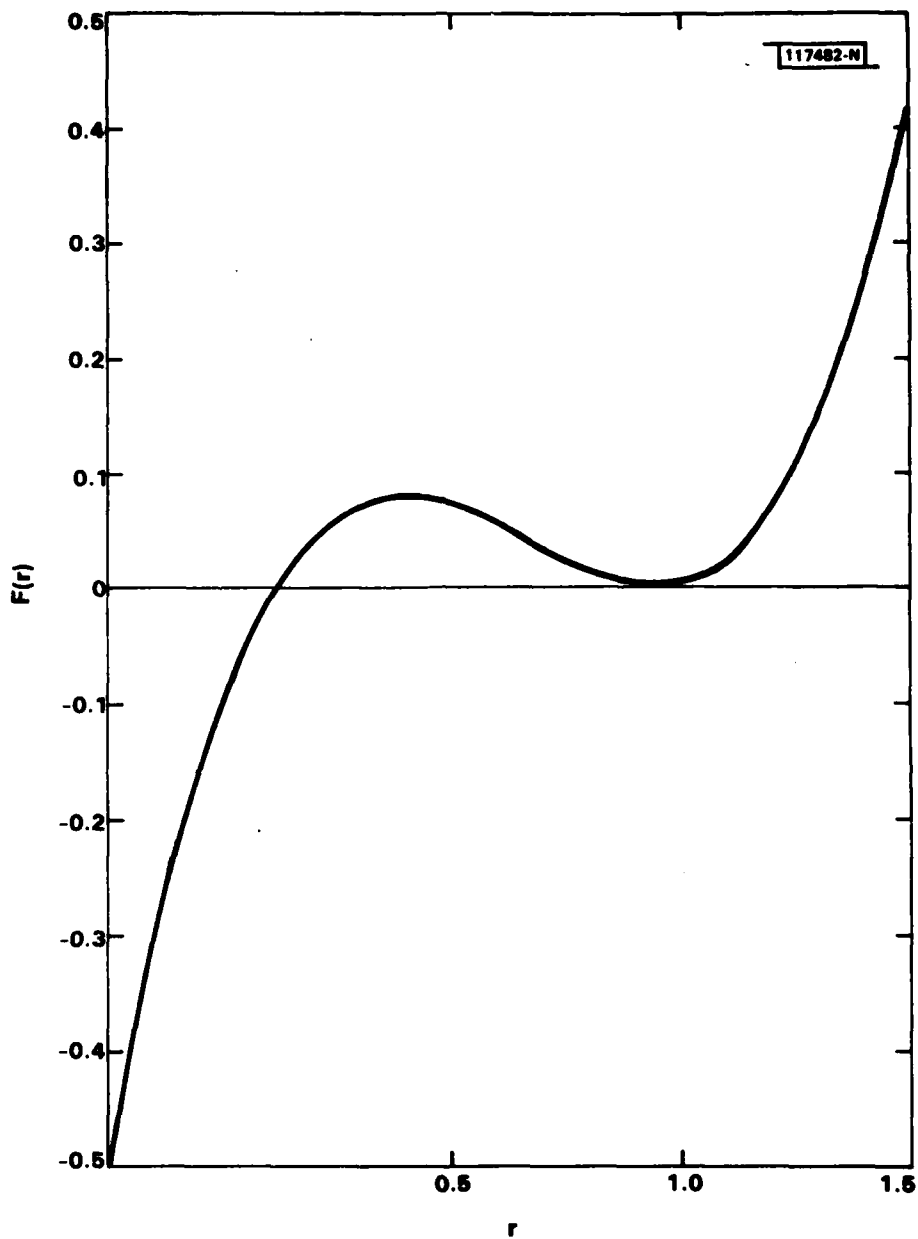
Fig. 3 F(r) near r = 0 for a = 1, e^2 = -1/9, J_2 = 0.107827329.
Note that the double root corresponds to a local minimum of F
and that this implies that the orbit is stable.

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IV. THE TIME DEPENDENCE OF ϕ AND r

A. The Exact Solution for ϕ and the Period

By combining Eqs. (4a, 16) we have

$$\dot{\phi} = (L/R_p^2)[1 - m\,sn^2(u,k)]nd^4(u,k)$$

where $u = \gamma\phi/2$ and $nd(\psi,\kappa) = 1/dn(\psi,\kappa)$. The initial condition is $\phi = 0$ when $t = T$. If we introduce the incomplete elliptic integrals of the second and third kinds, $E(\psi,\kappa)$ and $\Pi(\psi,n,\kappa)$, then we can integrate the above to obtain

$$2\pi(t-T)/P = \left\{ [R_0^2 - R_a R_p + R_0(R_a + R_p)]F(\beta,k) + R_p(R_a - R_0)E(\beta,k) \right.$$

$$\left. + (R_p - R_0)(R_a + R_p + R_0)\Pi(\beta,m,k) - \frac{R_p(R_a - R_p)\sin\beta\cos\beta(1 - k^2\sin^2\beta)^{1/2}}{1 - m\sin^2\beta} \right\} \Bigg/$$

$$[a^2 R_p(R_a - R_0)]^{1/2} \tag{21}$$

Here $P^2 \equiv 4\pi^2 a^3/\mu$ (e.g. P = the period were $J_2 = 0$). The angle β is related to ϕ via $\sin\beta = sn(u,k)$ or $\beta = am(u,k)$ where $am(\psi,\kappa)$ is the amplitude function of argument ψ, modulus κ.

Now if $t = T + t_a$ then $\phi = \phi_a$ and $u = \gamma\phi_a/2 = K(k)$. Note that $sn[K(k)k] = 1$ so $\beta = \pi/2$. As the incomplete elliptic integrals $F(\psi,\kappa)$, $E(\psi,\kappa)$, and $\Pi(\psi,n,\kappa)$ become the corresponding complete elliptic integrals $K(\kappa)$, $E(\kappa)$, and $\Pi(n,\kappa)$ when their argument is $\pi/2$, we find

$$2\pi t_a/P = \left\{ [R_0^2 - R_a R_p + R_0(R_a + R_p)]K(k) + R_p(R_a - R_0)E(k) + \right.$$

$$\left. (R_p - R_0)(R_a + R_p + R_0)\Pi(m,k) \right\} \Bigg/ [a^2 R_p(R_a - R_0)]^{1/2} \tag{22}$$

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Similarly, if $t = T + t_p$ then $\phi = \phi_p$, $u = \gamma\phi_p/2 = 2K(k)$. But $sn[2K(k),k] = 0$ so $\beta = \pi$. As all of the incomplete elliptic integrals obey the addition theorem $I(\pi,\kappa) = 2I(\pi/2,\kappa)$, it follows immediately that $t_p = 2t_a$. Therefore, not only is the orbit periodic but the time development of ϕ is periodic too. Furthermore the period is t_p and it requires as much time for ϕ to increase from 0 to ϕ_a as it does for ϕ to increase from ϕ_a to ϕ_p. For $a = 1$, $e = 1/3$, $J_2 = 1/10$, $2t_p/P = 0.976979$.

Our next quest is the value of t_p as $J_2 \rightarrow 0$. From Eq. (22) and the fact that as $\kappa^2 \rightarrow 0$

$$K(\kappa) \rightarrow \frac{\pi}{2} (1 + \kappa^2/4)$$

$$E(\kappa) \rightarrow \frac{\pi}{2} (1 - \kappa^2/4)$$

$$\Pi(n,\kappa) \rightarrow \frac{\pi}{2} (1 - n)^{-\frac{1}{2}} \left\{ 1 + (\kappa^2/2)[1 + (1 - n)^{-\frac{1}{2}}]^{-1} \right\}$$

one can show, laboriously, that

$$t_p = 2t_a \rightarrow P + \text{terms of order } J_2^2$$

Since $P^2 = 4\pi^2 a^3/\mu$ it follows that there is no secular change in a, due to J_2, to first order in J_2. This is a standard result of perturbation theory. The same statement is true of the eccentricity and can be derived herein from the constancy of L (by definition) and the absence of a first order term in P.

B. The Exact Solution for r

We can rewrite Eq. (16) as

$$\dot{r}^2 = - \left(\frac{2\pi a}{P}\right)^2 \frac{rF(r)}{r^4}$$

Remembering the discussion in § III A we see that

$$\int_{R_p}^{r} \frac{s^2 ds}{[-sF(s)]^{1/2}} = \frac{2\pi a}{P} \int_{T}^{t} d\tau \qquad t - T\epsilon[0,t_a]$$

$$\int_{R_a}^{r} \frac{s^2 ds}{[-sF(s)]^{1/2}} = \frac{-2\pi a}{P} \int_{T+t_a}^{t} d\tau \qquad t - T\epsilon[t_a,t_p]$$

To integrate this we need the following result: Let P(u) be the quartic polynomial $P = a_0 u^4 + a_1 u^3 + a_2 u^2 + a_3 u + a_4$, $a_0 \neq 0$ and let k be any integer >1, then

$$2(k-1)a_0 \int^{v} \frac{u^k du}{[P(u)]^{1/2}} = 2v^{k-3} P^{1/2}(v) + \sum_{j=1}^{4} (2+j-2k)a_j \int^{v} \frac{u^{k-j} du}{[P(u)]^{1/2}}$$

Here $P(s) = -sF(s)$, $a_0 = -1$, $a_1 = 2a$, $a_2 = -a^2(1-e^2)$, $a_3 = aJ_2$, $a_4 = 0$ and $k = 2$. Therefore one finds,

$$\frac{2\pi(t-T)}{P} = \left| 2\left[(R_p-R_0)\Pi(\lambda,m,k) + R_0 F(\lambda,k) \right] + (R_a/a)[(R_p-R_0). \right.$$

$$\left. \Pi(\lambda,k^2,k) - R_p F(\lambda,k)] \right| / [R_p(R_a-R_0)]^{1/2} - \left[\frac{-F(r)}{a^2 r} \right]^{1/2}$$

for $t - T\epsilon[0,t_a]$ and

$$\frac{2\pi(t-T-t_a)}{P} = \left| 2R_a\Pi(\Lambda,m',k) - [J_2/(k^2 R_a)][(k^2-m')F(\Lambda,k) + m'E(\Lambda,k)] \right|$$

$$/ [R_p(R_a-R_0)]^{1/2} + \left[\frac{-F(r)}{a^2 r} \right]^{1/2}$$

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for $t - T\epsilon[t_a, t_p]$ where $m' = -(m-k^2)/(1-m) = 1-R_a/R_p$. If one so desired one could show that $2t_a = t_p$ again and the absence of first order term in t_p/P as $J_2 \to 0$. Just as in the $J_2 = 0$ problem these two expressions are not identical; it's $r(t)$, $\phi(t)$, and $r(\phi)$ which hold over both parts of the orbit not their inverses. In particular, using many of the addition theorems for the elliptic integrals, the lower form may be written as

$$
\frac{2\pi(t-T-t_a)}{P} = 2[(R_p-R_0)[\Pi(m,k) - \Pi(\lambda,m,k)] + R_0[K(k) - F(\lambda,k)]
$$

$$
+ (R_a/a)[(R_p-R_0)[\Pi(k^2,k) - \Pi(\lambda,k^2,k)] - R_p[K(k) - F(\lambda,k)]\Big/
$$

$$
[R_p(R_a-R_0)]^{1/2} + \left[\frac{-F(r)}{a^2 r}\right]^{1/2} + 2\left\{\tan^{-1}\left[\frac{-F(r)}{r(r-R_0)^2}\right]^{1/2} - \right.
$$

$$
\left. \tan^{-1}\left[\frac{-F(r)m^{-2}}{r(r-R_0)^2}\right]^{1/2}\right\}
$$

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This Report discusses motion in the equatorial plane of an oblate primary. We include terms due to non-sphericity through J_2 in the potential. This problem is exactly soluble in the same sense that the ordinary two-body problem is exactly soluble. The full solution is given, the connection with perturbation theory (as $J_2 \rightarrow 0$) is demonstrated, and the stability of circular orbits is discussed. The analytical solution involves all three incomplete elliptic integrals which limits its transparency.

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